Inelastic Scattering and Interactions of Three-Wave Parametric Solitons

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Abstract

We study the excitation, decay and interactions of novel, velocity locked three-wave parametric solitons in a medium with quadratic nonlinearity and dispersion. We analytically describe the particle-like scattering between stable or unstable soliton triplets with linear waves in terms of explicit solutions featuring accelerated or decelerated solitons.

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Three-wave resonant interactions (TWRI) are widespread in various branches of physics, as they describe the resonant mixing of waves in weakly nonlinear and dispersive media. The TWRI model is typically encountered in the description of any conservative nonlinear medium where the nonlinear dynamics can be considered as a perturbation of the linear waves solution, the lowest-order nonlinearity is quadratic in the field amplitudes and the phase-matching (or resonance) condition is satisfied. Solutions to the TWRI have been known for a long time [1, 2, 3, 4, 5, 6, 7], and extensive applications are found in nonlinear optics (parametric amplification, frequency conversion, stimulated Raman and Brillouin scattering), plasma physics (laser-plasma interactions, radio frequency heating, plasma instabilities), acoustics (light-acoustic interactions), fluid dynamics (interaction of water waves) and solid state physics (wave-wave scattering). Soliton solutions of TWRI are of particular interest in the study of coherent energy transport and frequency conversion. Indeed, solitons behave as particles: as a result, different waves belonging to the same soliton may propagate locked together as a single entity. Such effect has no counterpart for linear waves [1, 8, 9, 10, 11, 12]. In a related paper [13], we discovered a novel multi-parameter TWRI soliton family consisting of a triplet of bright-bright-dark waves (or simulton) that travel with a common velocity. We could also identify the conditions for stable (fast) and unstable (slow) solitons. The fate of the unstable solitons however remained unknown.

In this Letter, the solution of this problem is fully reported by giving, strikingly enough, an analytic exact solution which describes the entire evolution of the unstable solitons. In fact, we reveal and explore a novel consequence of the particle-like nature of TWRI simultons (TWRIS), namely their inelastic scattering with particular linear waves. Such phenomenon is associated with the excitation (decay) of stable (unstable) simultons by means of the absorption (emission) of the energy carried by an isolated linear pulse. The decay (excitation) of simultons is associated with their speed-up (slowing-down) and creation of another triplet with complementary stability properties. As shown below, such processes are exactly described in terms of an analytical higher-order soliton solution with varying speed, or boomeron. The present TWRIS scattering process may be pictured as the interaction of radiation with a two-level atomic system: transitions among excited and ground soliton states are induced by the absorption and spontaneous emission of a wave.

The coupled partial differential equations that rule TWRI in $(1 + 1)$ dimensions read as

[2]:

$$E_{1t} - V_1 E_{1z} = \quad E_2^* E_3^*,$$
$$E_{2t} - V_2 E_{2z} = -E_1^* E_3^*, \qquad (1)$$
$$E_{3t} - V_3 E_{3z} = \quad E_1^* E_2^*,$$

where the subscripts t and z denote derivatives in the longitudinal and transverse dimensions, $E_n = E_n(z,t)$ are the complex wave amplitudes with velocities V_n, and $n = 1, 2, 3$. We chose here $V_1 > V_2 > V_3$ which, together with the above choice of the signs before the quadratic terms, entails the non-explosive character of the three-wave interaction [7]. In the following, with no loss of generality, we shall consider Eqs.(1) in a reference frame with $V_3 = 0$. Since we consider resonant interactions, the frequencies and momenta of the three waves must satisfy the prescriptions $\omega_1 + \omega_2 + \omega_3 = 0$ and $k_1 + k_2 + k_3 = 0$.

The TWRI equations (1) represent an infinite-dimensional Hamiltonian system, which conserves the Hamiltonian, the sum of the energies of waves E_1 and E_2, the sum of the energies of waves E_2 and E_3, and the total transverse momentum (see Ref. [13] for details).

Equations (1) exhibit a three-parameter family of simulton solutions in the form of bright-bright-dark triplets that travel with a common, or locked velocity V [13]. The most remarkable physical property of these simultons is that their speed V may be continuously varied by means of adjusting the energy of the two bright pulses. The propagation stability analysis of TWRIS reveals that a triplet is no longer stable whenever its velocity V decreases below a well defined (critical) value, namely $V < V_{cr} = 2V_1V_2/(V_1 + V_2)$ [13]. As an example, in Fig. 1(a) we show the contour plot of the amplitude of the three waves that compose an unstable simulton. These plots should be compared with Fig. 1(b), obtained from the numerical propagation with an initial (i.e., at $t = -5$) condition given by the exact simulton solution of Fig. 1(a). The results of Fig. 1 illustrate that an unstable simulton with $V < V_{cr}$ decays into a stable simulton with $V > V_{cr}$. This process is accompanied by the emission of an isolated pulse in wave E_3. It is quite remarkable that the simulton decay and wave emission as it is numerically observed in Fig. 1(b) may be exactly reproduced in terms of an analytical higher-order soliton solution with varying speed, or boomeron. Such solution was found by means of the techniques described in Ref.[14], and it can be expressed as

$$E_1 = \frac{2pV_2}{\Delta}\sqrt{\frac{2V_1}{V_1 - V_2}}e^{iq_1z_1}(H_+^* e^{-i\theta} - H_-^* e^{i\theta}), \qquad (2a)$$

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$$E_2 = \frac{2pV_1}{\Delta}\sqrt{\frac{2V_2}{V_1 - V_2}}e^{iq_2 z_2}\left(\sqrt{(1-Q)/(1+Q)}\right.$$
$$\left. H_+ e^{i(\beta+\theta)} - \sqrt{(1+Q)/(1-Q)}H_- e^{-i(\beta+\theta)}\right), \tag{2b}$$

$$E_3 = a\sqrt{V_1 V_2}e^{iq_3 z_3} - \frac{\Delta}{4p}\left(\frac{V_1 - V_2}{V_1 V_2}\right)E_1^* E_2^*, \tag{2c}$$

where

$$\Delta = 1 + \frac{|H_+|^2}{1+Q} + \frac{|H_-|^2}{1-Q} - 2\cos(\beta)\mathcal{R}e(H_+ H_-^* e^{i(\beta+2\theta)}),$$

$$H_\pm(z,t) = e^{(-B_\pm + i\chi_\pm)z}\, e^{\frac{-2V_1 V_2}{V_1 - V_2}(p-ik)t},$$

$$\omega = -2k\frac{V_1 V_2}{V_1 - V_2}, \quad \chi_\pm = k\left(\frac{V_1 + V_2}{V_1 - V_2}\mp\frac{1}{Q}\right),$$

$$B_\pm = p\left(\frac{V_1 + V_2}{V_1 - V_2}\mp Q\right), \quad \tan(\beta) = k/(pQ),$$

$$Q = \frac{1}{p}\sqrt{\frac{1}{2}[\,r + \sqrt{r^2 + 4k^2 p^2}\,]}, \ r = p^2 - k^2 - a^2,$$

$$q_n = q(V_{n+1} - V_{n+2}),$$

$$z_n = z + V_n t, \ n = 1, 2, 3\,mod(3).$$

It is worth noting that the above solution depends upon seven real parameters $V_1, V_2, p, k, q, a, \theta$. From the definition of Q, it is apparent that these parameters must be chosen in such a way that if $k = 0$, then $p^2 > a^2$.

The analytical solution (2), while rather complicate at intermediate times, asymptotically consists of one or two coherent structures. In fact, let us consider first the decay process: if we assume $p < 0$, for negative large t ($t \to -\infty$) the boomeron is asymptotically composed of two bright pulses (E_1, E_2) and a kink-like pulse (E_3) travelling with the locked velocity V_i. If instead t is large and positive ($t \to +\infty$) the boomeron is composed of two bright pulses (E_1, E_2) and a kink-like pulse (E_3) traveling at the locked velocity V_f ($V_f > V_i$), plus another pulse (E_3) that travels with the linear group velocity V_3. The velocities V_f and V_i can be calculated from (2):

$$V_i = \frac{2V_1 V_2}{V_1 + V_2 - Q(V_1 - V_2)}, \tag{3}$$

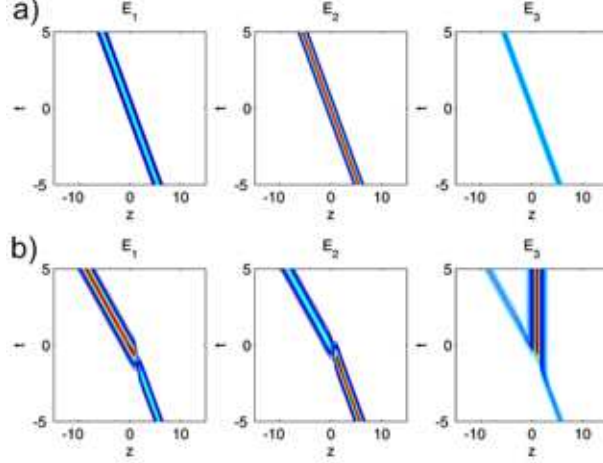$$V_f = \frac{2V_1 V_2}{V_1 + V_2 + Q(V_1 - V_2)}. \tag{4}$$

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FIG. 1: a) Analytical solution; b) numerical propagation of an unstable TWRIS, which coincides with a boomeron solution. Here $V_1 = 2, V_2 = 1$. The simulton velocity is $V = 1.1$ ($V < V_{cr} \approx 1.3$).

The triplet travelling at very large $|t|$ with the locked velocity V_i (V_f) is itself an exact solution of Eqs.(1), namely it is the unstable (stable) TWRIS as presented in Ref.[13]. Therefore the boomeron solution (2) provides the exact description of the decay from unstable into stable solitons.

Let us consider next the situation where a stable TWRIS collides with an isolated pulse in the wave E_3, namely the excitation by absorption. Once again, this scattering process is exactly described by the boomeron solution (2), and it leads to the excitation of an unstable TWRIS, induced by the absorption of the isolated wave E_3. Indeed, whenever $p > 0$ and t is very large and negative, the boomeron (2) is composed of a triplet consisting of two bright pulses (in waves E_1, E_2) and a kink-like pulse (in wave E_3), all traveling with the same velocity V_i, plus an isolated pulse in wave (E_3) that travels with the linear group velocity V_3. The triplet and the isolated pulse collide and, as a result, the pulse in E_3 is completely absorbed by the triplet. Finally, for very large and positive t the boomeron consists of a single triplet formed by two bright pulses (in waves E_1, E_2) and a kink-like pulse (in wave E_3), again traveling together with the velocity V_f ($V_f < V_i$). Note that the asymptotic boomeron triplets traveling with velocities V_i and V_f can be analytically mapped into the stable and unstable TWRIS as given in [13]. In conclusion, the analytical solution (2) with $p > 0$ provides the exact description of the excitation of an unstable TWRIS as a result of the inelastic collision between a stable TWRIS and a linear wave packet.
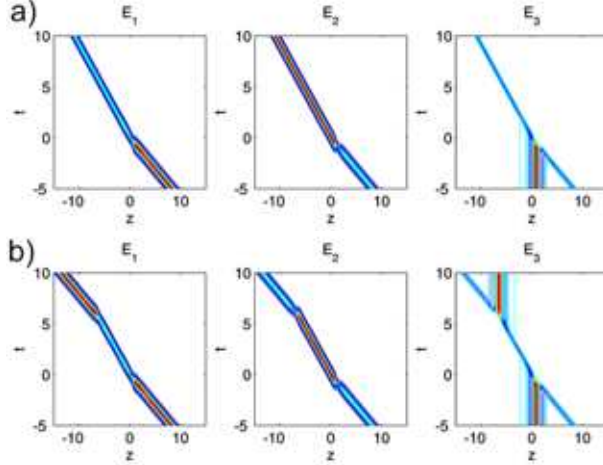
FIG. 2: a) Analytical boomeron solution describing the collision of a stable TWRIS with a single pulse in wave E_3. Parameters are $V_1 = 2, V_2 = 1, V_3 = 0, p = 1, a = 1, k = 0.5, q = 1, \theta = \pi/6$. The triplet velocities are $V_i = 1.8$ and $V_f = 1.1$ ($V_{cr} \approx 1.3$). b) Numerical double scattering process.

Figure 2(a) displays the analytical boomeron solution corresponding to the collision between a stable TWRIS and a pulse in wave E_3. Whereas Fig. 2(b) shows the inelastic scattering of the TWRIS and the linear wave as numerically computed by integrating the equations (1) with the initial data at $t = -0.5$ equal to the solution of Fig. 2(a). As it can be seen in Fig. 2(b), the excited unstable TWRIS has a finite lifetime since it eventually decays into a stable or ground state TWRIS via the emission of another linear wave. It is worth noting that both the excitation and the decay processes may described by properly adjusting the parameters of Eqs.(2).

The dynamics of the scattering between TWRIS and linear waves is analogous to the interaction between radiation and a two-level atom. Indeed, transitions between excited and ground soliton states are induced by the absorption and spontaneous emission of a linear pulse in the wave E_3.

Let us now briefly discuss the role of the various parameters in Eqs. (2). Two of these parameters (i.e. the velocities V_1 and V_2) are fixed by the linear dispersive properties of the medium. We are thus left with five independent real parameters, namely p, k, q, a, θ (with the restrictions $a > 0$ and $0 \le \theta < 2\pi$). We point out that our discussion above implies that the specification of these parameters allows one to define the properties of both unstable and stable TWRIS since these solitons result as asymptotic states of the analytic boomeron
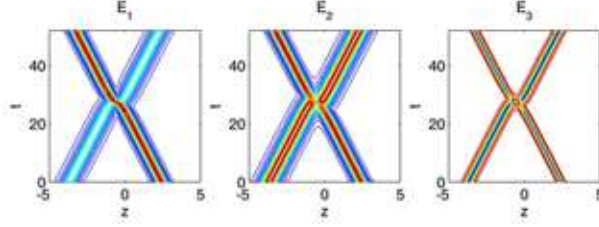
FIG. 3: Collision of two stable TWRIS with different velocities. Fast simulton $V = 1.9$, slow simulton $V = 1.7$. Simulation is performed in reference frame moving at velocity $V_{ref} = 1.8$.
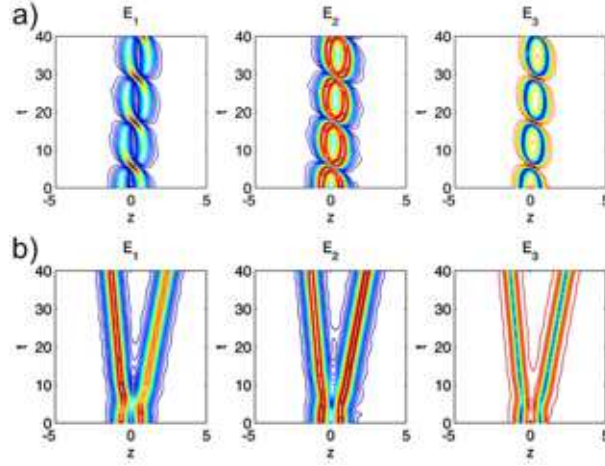

FIG. 4: a) Collision of two equal and in-phase stable TWRIS with the same velocity $V = 1.8$; b) collision of two equal and $\pi/4$ out-of-phase stable TWRIS with the same velocity. Simulations are performed in reference frame moving at velocity $V_{ref} = 1.8$.

expression (2) in the limit as $|t| \to \infty$. The parameter p is associated with the rescaling of the wave amplitudes, and of the coordinates z and t. Whereas a measures the amplitude of the kink background in wave E_3. The value of k is related to the soliton wavenumber. The parameter q simply adds a phase shift which is linear in both z and t. Finally, θ fixes the shape of the stationary kink pulse E_3. By adjusting the various degrees of freedom of the boomeron family of solutions (2), one may foresee the dynamical reshaping of the amplitude, phase, and velocity of the TWRIS, as well as fully describe the process of energy exchange among the three waves.

In order to emphasize the novel and striking features of the scattering between TWRIS and linear waves, let us briefly consider now, the collisions between different TWRIS. Since

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Eqs.(1) are completely integrable, interactions between two initially well-separated TWRIS do not modify the shapes of triplets that emerge after the collision. Indeed, the numerical simulation of Fig. 3 shows that two TWRIS with different velocities penetrate and cross each other with no change of their shapes. The only effect of the interaction is a spatial shift and a phase shift, as it happens with ordinary bright TWRI solitons [10]. However, in a manner similar to cubic nonlinear Schrödinger solitons ([15, 16], [17] and references therein), whenever the initial simulton separation is reduced, complex interaction phenomena may take place owing to the excitation of higher order soliton solutions. For example, Fig. 4(a) shows that two equal and in-phase TWRIS with the same velocity attract each other and periodically collapse. Whereas Fig. 4(b) shows that a repulsive force exists between two equal and out-of-phase solitons with the same velocity (the phase difference α between the two solitons is imposed by multiplying the wave E_1, respectively E_2, of one of the solitons by the phase factor $\exp[i\alpha]$, respectively $\exp[-i\alpha]$). In this case, two distinct TWRIS moving with different velocities emerge from the initial collision. Hence TWRI solitons may cross, attract or repel each other depending on their initial separation, velocity difference, and relative phase.

In conclusion, we described in terms of analytical solutions the scattering process of three-wave simultons and linear waves. An unstable simulton decays into a stable simulton by accelerating its speed and emitting an isolated pulse. Moreover, a stable triplet may be excited into an unstable simulton by slowing down as a result of the absorption of a linear wave. Finally, simultons with different speeds are stable upon collision, and simultons with equal speeds interact with each other in a way which is strongly dependent upon their initial relative phase.

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